

ON ADDITIVE MAPPINGS WITH BOUNDED VALUE NEAR A POINT

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ABSTRACT. In this paper, we will first prove the linearity of additive mappings with bounded value near a point. And by using our theorem we will investigate the instability of the Cauchy equation in \mathbb{R}^2 .

1. Introduction

In 1821, Cauchy [1] proved the following theorem.

PROPOSITION 1.1. *If an additive mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then f is linear.*

In 1946, Kestelman [3] proved the following theorem.

PROPOSITION 1.2. *Let E_1 and E_2 be normed spaces, and let E be a subset of E_1 . Suppose that there is a $\delta > 0$ satisfying*

$$\{v \in E_1 : \|v\| < \delta\} \subset \{x - y : x, y \in E\}.$$

If an additive mapping $f : E_1 \rightarrow E_2$ is bounded on E , then f is linear.

In 1991, Gajda [2] also proved the following theorem.

PROPOSITION 1.3. *Let E_1 be a normed space and E_2 a Banach space. And let $\varepsilon > 0$ and $p \in \mathbb{R} \setminus \{1\}$. If $f : E_1 \rightarrow E_2$ is a mapping such that*

$$\|f(x + y) - f(x) - f(y)\| < \varepsilon(\|x\|^p + \|y\|^p)$$

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for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{1-p}{|1-p|} \frac{2\varepsilon}{2-2^p} \|x\|^p$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then T is linear.

In 2010, Reem [4] proved the following theorem.

PROPOSITION 1.4. *If an additive mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ is integrable on some bounded closed interval I , then f is linear.*

In this paper, we will present another version and proof of Proposition 1.2 as follows.

THEOREM 1.5. *Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a mapping satisfying the Cauchy equation*

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. If T is bounded mapping on a small disk $B(\mathbf{x}, r) = \{\mathbf{z} \in \mathbb{R}^2 \mid |\mathbf{z} - \mathbf{x}| < r\}$ for some $\mathbf{x} \in \mathbb{R}^2$, then there exists a 2×2 matrix A such that $T(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^2$. Therefore, the mapping $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear mapping.

Gajda [2] investigated the instability of the Cauchy equation in \mathbb{R} as follows.

PROPOSITION 1.6. *For any $\varepsilon > 0$, there exists a mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying*

$$|f(x+y) - f(x) - f(y)| < \varepsilon(|x| + |y|)$$

for all $x, y \in \mathbb{R}$ such that there is no constant $\delta \geq 0$ and no additive function $T : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the condition

$$|f(x) - T(x)| \leq \delta|x|$$

for all $x \in \mathbb{R}$.

In this paper, we will also investigate the instability of the Cauchy equation in \mathbb{R}^2 as follows.

THEOREM 1.7. *For any $\varepsilon > 0$, there exists a mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying*

$$|f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon(|\mathbf{x}| + |\mathbf{y}|)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that there is no constant $\delta \geq 0$ and no additive function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying the condition

$$|f(\mathbf{x}) - T(\mathbf{x})| \leq \delta|\mathbf{x}|$$

for all $\mathbf{x} \in \mathbb{R}^2$. So we can say about the instability of the Cauchy equation in \mathbb{R}^2 .

2. Main Results

In this section, we will prove main results.

THEOREM 2.1. *Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a mapping satisfying the Cauchy equation*

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. If T is bounded on a small disk $B(\mathbf{x}, r) = \{\mathbf{z} \in \mathbb{R}^2 \mid |\mathbf{z} - \mathbf{x}| < r\}$ for some $\mathbf{x} \in \mathbb{R}^2$, then there exists a 2×2 matrix A such that $T(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^2$. Therefore, the mapping $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear mapping.

Proof : If $T = (T_1, T_2)$ is bounded on a small disk $B(\mathbf{x}, r) = \{\mathbf{z} \in \mathbb{R}^2 \mid |\mathbf{z} - \mathbf{x}| < r\}$, then T is bounded on a small disk containing $\mathbf{0} \in \mathbb{R}^2$. Since T is bounded on a small disk containing $\mathbf{0} \in \mathbb{R}^2$ there exists $\mathbf{z}_0 = (\pi_1(\mathbf{z}_0), \pi_2(\mathbf{z}_0))$ such that $\pi_1(\mathbf{z}_0) \neq 0$, $\pi_2(\mathbf{z}_0) \neq 0$, where $\pi_1, \pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ are the projection mappings given by $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$, respectively.

Define $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$(2.1) \quad G(\mathbf{z}) := T \begin{pmatrix} \pi_1(\mathbf{z}) \\ \pi_2(\mathbf{z}) \end{pmatrix} - \begin{pmatrix} \frac{1}{\pi_1(\mathbf{z}_0)} T_1 \begin{pmatrix} \pi_1(\mathbf{z}_0) \\ 0 \end{pmatrix} & \frac{1}{\pi_2(\mathbf{z}_0)} T_1 \begin{pmatrix} 0 \\ \pi_2(\mathbf{z}_0) \end{pmatrix} \\ \frac{1}{\pi_1(\mathbf{z}_0)} T_2 \begin{pmatrix} \pi_1(\mathbf{z}_0) \\ 0 \end{pmatrix} & \frac{1}{\pi_2(\mathbf{z}_0)} T_2 \begin{pmatrix} 0 \\ \pi_2(\mathbf{z}_0) \end{pmatrix} \end{pmatrix} \begin{pmatrix} \pi_1(\mathbf{z}) \\ \pi_2(\mathbf{z}) \end{pmatrix}$$

for all $\mathbf{z} \in \mathbb{R}^2$. Then, by the additive property of T , the mapping G is also additive. And, we have $G(\mathbf{z}_0) = \mathbf{0}$ and $G(\mathbf{z} + \mathbf{z}_0) = G(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^2$.

Moreover, we have

$$G \left(\mathbf{z} + \begin{pmatrix} \pi_1(\mathbf{z}_0) \\ 0 \end{pmatrix} \right) = G(\mathbf{z}) \quad \text{and} \quad G \left(\mathbf{z} + \begin{pmatrix} 0 \\ \pi_2(\mathbf{z}_0) \end{pmatrix} \right) = G(\mathbf{z})$$

for all $\mathbf{z} \in \mathbb{R}^2$. Hence the additive mapping $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a periodic mapping which implies that G is a bounded mapping on \mathbb{R}^2 . So, if there exists $\mathbf{w} \in \mathbb{R}^2$ such that $|G(\mathbf{w})| \neq \mathbf{0}$ then we can obtain

$$|G(n\mathbf{w})| = |nG(\mathbf{w})| \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty$$

which contradicts the bounded property of G . Therefore, we have $G(\mathbf{z}) = \mathbf{0}$ for all $\mathbf{z} \in \mathbb{R}^2$. As a result we can have $T(\mathbf{z}) = A\mathbf{z}$, where

$$A = \begin{pmatrix} \frac{1}{\pi_1(\mathbf{z}_0)} T_1 \begin{pmatrix} \pi_1(\mathbf{z}_0) \\ 0 \end{pmatrix} & \frac{1}{\pi_2(\mathbf{z}_0)} T_1 \begin{pmatrix} 0 \\ \pi_2(\mathbf{z}_0) \end{pmatrix} \\ \frac{1}{\pi_1(\mathbf{z}_0)} T_2 \begin{pmatrix} \pi_1(\mathbf{z}_0) \\ 0 \end{pmatrix} & \frac{1}{\pi_2(\mathbf{z}_0)} T_2 \begin{pmatrix} 0 \\ \pi_2(\mathbf{z}_0) \end{pmatrix} \end{pmatrix}.$$

Hence $T(\mathbf{z}) = A\mathbf{z}$ is a linear transformation and we complete the proof. \square

THEOREM 2.2. *For any $\varepsilon > 0$, there exists a mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying*

$$(2.2) \quad |f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon(|\mathbf{x}| + |\mathbf{y}|)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that there is no constant $\delta \geq 0$ and no additive function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying the condition

$$|f(\mathbf{x}) - T(\mathbf{x})| \leq \delta|\mathbf{x}|$$

for all $\mathbf{x} \in \mathbb{R}^2$. So, we can say about the instability of the Cauchy equation in \mathbb{R}^2 .

Proof : Let $\mu := \frac{\varepsilon}{6\sqrt{2}}$ and let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$\phi(x) := \begin{cases} \mu x & \text{for } |x| < 1 \\ \mu \frac{x}{|x|} & \text{for } |x| \geq 1. \end{cases}$$

Note that $|\phi(x)| \leq \mu$ for all $x \in \mathbb{R}$. Define $f = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$f(x, y) = (f_1(x, y), f_2(x, y)) := \left(\sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n y)}{2^n} \right)$$

for all $(x, y) \in \mathbb{R}^2$, then $f(0, 0) = (0, 0)$. It is easily shown that

$$\begin{aligned} |f(x, y)|^2 &= f_1(x, y)^2 + f_2(x, y)^2 \leq \left(\sum_{n=0}^{\infty} \frac{\mu}{2^n} \right)^2 + \left(\sum_{n=0}^{\infty} \frac{\mu}{2^n} \right)^2 \\ &= 2\mu^2 \left(\sum_{n=0}^{\infty} \frac{1}{2^n} \right)^2 = 8\mu^2 \end{aligned}$$

for all $(x, y) \in \mathbb{R}^2$. That is, $|f(x, y)| \leq 2\sqrt{2}\mu$ for all $(x, y) \in \mathbb{R}^2$.

Now, we shall prove that there is a $\varepsilon > 0$ such that

$$(2.3) \quad |f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)| \leq \varepsilon(|(x_1, x_2)| + |(y_1, y_2)|)$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$.

- CASE 1°: If $(x_1, x_2) = (y_1, y_2) = (0, 0)$, then (2.3) is fulfilled.
- CASE 2°: Assume that $0 < |(x_1, x_2)| + |(y_1, y_2)| < 1$. Then there exists $N \in \mathbb{N}$ such that

$$(2.4) \quad \frac{1}{2^N} < |(x_1, x_2)| + |(y_1, y_2)| < \frac{1}{2^{N-1}}.$$

Hence, we see that

$$\begin{aligned} 2^{N-1}|(x_1, x_2)| &< 1, \quad 2^{N-1}|(y_1, y_2)| < 1, \\ 2^{N-1}|(x_1, x_2) + (y_1, y_2)| &\leq 2^{N-1}(|(x_1, x_2)| + |(y_1, y_2)|) < 1. \end{aligned}$$

Note that for $n = 0, 1, \dots, N - 1$ we have $2^n \leq 2^{N-1}$. So we get the followings :

$$\begin{aligned} |2^n x_1|, |2^n x_2| &\leq |2^n(x_1, x_2)| = 2^n|(x_1, x_2)| < 2^{N-1}|(x_1, x_2)| < 1, \\ |2^n y_1|, |2^n y_2| &\leq |2^n(y_1, y_2)| = 2^n|(y_1, y_2)| < 2^{N-1}|(y_1, y_2)| < 1 \end{aligned}$$

and

$$\begin{aligned} |2^n(x_1 + y_1)|, |2^n(x_2 + y_2)| &\leq |2^n(x_1 + y_1), 2^n(x_2 + y_2)| \\ &= 2^n|(x_1, x_2) + (y_1, y_2)| < 2^{N-1}|(x_1, x_2) + (y_1, y_2)| < 1. \end{aligned}$$

Therefore, we have

$$2^n x_1, 2^n x_2, 2^n y_1, 2^n y_2, 2^n(x_1 + y_1), 2^n(x_2 + y_2) \in (-1, 1).$$

Since ϕ is linear on the interval $(-1, 1)$, we find that

$$(2.5) \quad \begin{aligned} \phi(2^n(x_1 + y_1)) - \phi(2^n x_1) - \phi(2^n y_1) &= 0, \\ \phi(2^n(x_2 + y_2)) - \phi(2^n x_2) - \phi(2^n y_2) &= 0 \end{aligned}$$

for $n = 0, 1, \dots, N - 1$.

Now, we have

$$\begin{aligned} &\frac{|f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)|}{|(x_1, x_2)| + |(y_1, y_2)|} \\ &= \frac{1}{|(x_1, x_2)| + |(y_1, y_2)|} \cdot \left| \left(\sum_{n=0}^{\infty} \frac{\phi(2^n(x_1 + y_1))}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n(x_2 + y_2))}{2^n} \right) \right. \\ &\quad \left. - \left(\sum_{n=0}^{\infty} \frac{\phi(2^n x_1)}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n x_2)}{2^n} \right) - \left(\sum_{n=0}^{\infty} \frac{\phi(2^n y_1)}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n y_2)}{2^n} \right) \right|. \end{aligned}$$

Thus we obtain that

$$\begin{aligned}
& \frac{|f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)|}{|(x_1, x_2)| + |(y_1, y_2)|} \\
&= \frac{1}{|(x_1, x_2)| + |(y_1, y_2)|} \\
&\quad \cdot \left| \left(\sum_{n=0}^{\infty} \frac{\phi(2^n(x_1 + y_1))}{2^n} - \sum_{n=0}^{\infty} \frac{\phi(2^n x_1)}{2^n} - \sum_{n=0}^{\infty} \frac{\phi(2^n y_1)}{2^n}, \right. \right. \\
&\quad \left. \left. \sum_{n=0}^{\infty} \frac{\phi(2^n(x_2 + y_2))}{2^n} - \sum_{n=0}^{\infty} \frac{\phi(2^n x_2)}{2^n} - \sum_{n=0}^{\infty} \frac{\phi(2^n y_2)}{2^n} \right) \right| \\
&= \left| \frac{1}{|(x_1, x_2)| + |(y_1, y_2)|} \left(\sum_{n=0}^{\infty} \frac{\phi(2^n(x_1 + y_1)) - \phi(2^n x_1) - \phi(2^n y_1)}{2^n}, \right. \right. \\
&\quad \left. \left. \sum_{n=0}^{\infty} \frac{\phi(2^n(x_2 + y_2)) - \phi(2^n x_2) - \phi(2^n y_2)}{2^n} \right) \right| \\
&= \left| \left(\sum_{n=0}^{\infty} \frac{\phi(2^n(x_1 + y_1)) - \phi(2^n x_1) - \phi(2^n y_1)}{2^n (|(x_1, x_2)| + |(y_1, y_2)|)}, \right. \right. \\
&\quad \left. \left. \sum_{n=0}^{\infty} \frac{\phi(2^n(x_2 + y_2)) - \phi(2^n x_2) - \phi(2^n y_2)}{2^n (|(x_1, x_2)| + |(y_1, y_2)|)} \right) \right|.
\end{aligned}$$

By (2.5), we see that

$$\begin{aligned}
& \frac{|f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)|}{|(x_1, x_2)| + |(y_1, y_2)|} \\
&= \left| \left(\sum_{n=N}^{\infty} \frac{\phi(2^n(x_1 + y_1)) - \phi(2^n x_1) - \phi(2^n y_1)}{2^n (|(x_1, x_2)| + |(y_1, y_2)|)}, \right. \right. \\
&\quad \left. \left. \sum_{n=N}^{\infty} \frac{\phi(2^n(x_2 + y_2)) - \phi(2^n x_2) - \phi(2^n y_2)}{2^n (|(x_1, x_2)| + |(y_1, y_2)|)} \right) \right| \\
&= \left| \left(\sum_{k=0}^{\infty} \frac{\phi(2^{k+N}(x_1 + y_1)) - \phi(2^{k+N} x_1) - \phi(2^{k+N} y_1)}{2^{k+N} (|(x_1, x_2)| + |(y_1, y_2)|)}, \right. \right. \\
&\quad \left. \left. \sum_{k=0}^{\infty} \frac{\phi(2^{k+N}(x_2 + y_2)) - \phi(2^{k+N} x_2) - \phi(2^{k+N} y_2)}{2^{k+N} (|(x_1, x_2)| + |(y_1, y_2)|)} \right) \right|.
\end{aligned}$$

So, we obtain that

$$\begin{aligned} & \frac{|f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)|}{|(x_1, x_2)| + |(y_1, y_2)|} \\ &= \left(\left[\sum_{k=0}^{\infty} \frac{\phi(2^{k+N}(x_1 + y_1)) - \phi(2^{k+N}x_1) - \phi(2^{k+N}y_1)}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right]^2 \right. \\ & \quad \left. + \left[\sum_{k=0}^{\infty} \frac{\phi(2^{k+N}(x_2 + y_2)) - \phi(2^{k+N}x_2) - \phi(2^{k+N}y_2)}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right]^2 \right)^{\frac{1}{2}} \\ &= \left(\left| \sum_{k=0}^{\infty} \frac{\phi(2^{k+N}(x_1 + y_1)) - \phi(2^{k+N}x_1) - \phi(2^{k+N}y_1)}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right|^2 \right. \\ & \quad \left. + \left| \sum_{k=0}^{\infty} \frac{\phi(2^{k+N}(x_2 + y_2)) - \phi(2^{k+N}x_2) - \phi(2^{k+N}y_2)}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Note that

$$\begin{aligned} & \left| \sum_{k=0}^{\infty} \frac{\phi(2^{k+N}(x_1 + y_1)) - \phi(2^{k+N}x_1) - \phi(2^{k+N}y_1)}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right|^2 \\ & \leq \left[\sum_{k=0}^{\infty} \left| \frac{\phi(2^{k+N}(x_1 + y_1)) - \phi(2^{k+N}x_1) - \phi(2^{k+N}y_1)}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right| \right]^2 \\ & = \left[\sum_{k=0}^{\infty} \frac{|\phi(2^{k+N}(x_1 + y_1)) - \phi(2^{k+N}x_1) - \phi(2^{k+N}y_1)|}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right]^2 \\ & \leq \left[\sum_{k=0}^{\infty} \frac{|\phi(2^{k+N}(x_1 + y_1))| + |\phi(2^{k+N}x_1)| + |\phi(2^{k+N}y_1)|}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right]^2 \\ & \leq 9\mu^2 \left[\sum_{k=0}^{\infty} \frac{1}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right]^2. \end{aligned}$$

By the same argument as above, we also obtain that

$$\begin{aligned} & \left| \sum_{k=0}^{\infty} \frac{\phi(2^{k+N}(x_2 + y_2)) - \phi(2^{k+N}x_2) - \phi(2^{k+N}y_2)}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right|^2 \\ & \leq 9\mu^2 \left[\sum_{k=0}^{\infty} \frac{1}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} \right]^2. \end{aligned}$$

Hence we obtain that

$$\begin{aligned} & \frac{|f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)|}{|(x_1, x_2)| + |(y_1, y_2)|} \\ &= \sum_{k=0}^{\infty} \frac{3\sqrt{2}\mu}{2^{k+N}(|(x_1, x_2)| + |(y_1, y_2)|)} < \sum_{k=0}^{\infty} \frac{3\sqrt{2}\mu}{2^k} = 6\sqrt{2}\mu = \varepsilon. \end{aligned}$$

• CASE 3°: We finally assume that $|(x_1, x_2)| + |(y_1, y_2)| \geq 1$. Then, we get

$$\begin{aligned} & \frac{|f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)|}{|(x_1, x_2)| + |(y_1, y_2)|} \\ & \leq |f((x_1, x_2) + (y_1, y_2)) - f(x_1, x_2) - f(y_1, y_2)| \\ & \leq 6\sqrt{2}\mu = \varepsilon. \end{aligned}$$

By CASE 1°, CASE 2° and CASE 3°, we have the inequality (2.3) for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$.

To complete the proof we assume that there exists a $\delta \geq 0$ and an additive mapping $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$|f(x_1, x_2) - T(x_1, x_2)| \leq \delta|(x_1, x_2)|, \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2,$$

and prove that the assumption is contradictory. Since f_1 and f_2 are defined by means of a uniformly convergent series of continuous functions, they are continuous. So f is continuous. Hence the mapping $T = (T_1, T_2)$ is bounded on a neighborhood of $(0, 0)$. Then, by Theorem 2.1 there exists $A \in M_2(\mathbb{R})$ such that

$$T(x_1, x_2) = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where

$$A = \begin{pmatrix} \frac{T_1(x_0, 0)}{x_0} & \frac{T_1(y_0, 0)}{y_0} \\ \frac{T_2(x_0, 0)}{x_0} & \frac{T_2(y_0, 0)}{y_0} \end{pmatrix}$$

for some (x_0, y_0) containing in a neighborhood of $(0, 0)$. Hence, we get

$$\left| f(x_1, x_2) - A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right| \leq \delta|(x_1, x_2)|, \quad (x_1, x_2) \in \mathbb{R}^2.$$

So, we see that

$$|f(x_1, x_2)| \leq \delta|(x_1, x_2)| + \left| A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right|.$$

This implies that

$$(2.6) \quad \frac{|f(x_1, x_2)|}{|(x_1, x_2)|} \leq \delta + \frac{|A(x_1)|}{|(x_1, x_2)|} \leq \delta + \|A\|$$

for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. On the other hand, by the Archimedean property, we can choose $N \in \mathbb{N}$ such that $N\mu > \delta + \|A\|$. So

$$2^n x_1 \in (0, 1) \text{ for } n = 0, 1, \dots, N - 1 \text{ if } x_1 \in \left(0, \frac{1}{2^{N-1}}\right),$$

$$2^n x_2 \in (0, 1) \text{ for } n = 0, 1, \dots, N - 1 \text{ if } x_2 \in \left(0, \frac{1}{2^{N-1}}\right).$$

For $x_1, x_2 \in \left(0, \frac{1}{2^{N-1}}\right)$, we get

$$\begin{aligned} & \frac{|f(x_1, x_2)|}{|(x_1, x_2)|} \\ &= \frac{1}{|(x_1, x_2)|} \left| \left(\sum_{n=0}^{\infty} \frac{\phi(2^n x_1)}{2^n}, \sum_{n=0}^{\infty} \frac{\phi(2^n x_2)}{2^n} \right) \right| \\ &= \left| \left(\sum_{n=0}^{\infty} \frac{\phi(2^n x_1)}{2^n |(x_1, x_2)|}, \sum_{n=0}^{\infty} \frac{\phi(2^n x_2)}{2^n |(x_1, x_2)|} \right) \right| \\ &= \left| \left(\sum_{n=0}^{N-1} \frac{\mu 2^n x_1}{2^n |(x_1, x_2)|} + \sum_{n=N}^{\infty} \frac{\phi(2^n x_1)}{2^n |(x_1, x_2)|}, \right. \right. \\ & \quad \left. \left. \sum_{n=0}^{N-1} \frac{\mu 2^n x_2}{2^n |(x_1, x_2)|} + \sum_{n=N}^{\infty} \frac{\phi(2^n x_2)}{2^n |(x_1, x_2)|} \right) \right|. \end{aligned}$$

Thus we have

$$\begin{aligned} & \frac{|f(x_1, x_2)|}{|(x_1, x_2)|} \\ &= \left(\left[\sum_{n=0}^{N-1} \frac{\mu 2^n x_1}{2^n |(x_1, x_2)|} + \sum_{n=N}^{\infty} \frac{\phi(2^n x_1)}{2^n |(x_1, x_2)|} \right]^2 \right. \\ & \quad \left. + \left[\sum_{n=0}^{N-1} \frac{\mu 2^n x_2}{2^n |(x_1, x_2)|} + \sum_{n=N}^{\infty} \frac{\phi(2^n x_2)}{2^n |(x_1, x_2)|} \right]^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Hence we have

$$\begin{aligned} & \frac{|f(x_1, x_2)|}{|(x_1, x_2)|} \\ & \geq \left(\left[\sum_{n=0}^{N-1} \frac{\mu 2^n x_1}{2^n |(x_1, x_2)|} \right]^2 + \left[\sum_{n=0}^{N-1} \frac{\mu 2^n x_2}{2^n |(x_1, x_2)|} \right]^2 \right)^{\frac{1}{2}} \\ & = \left(\left[\sum_{n=0}^{N-1} \frac{\mu x_1}{|(x_1, x_2)|} \right]^2 + \left[\sum_{n=0}^{N-1} \frac{\mu x_2}{|(x_1, x_2)|} \right]^2 \right)^{\frac{1}{2}} \\ & = \frac{N\mu}{|(x_1, x_2)|} (x_1^2 + x_2^2)^{\frac{1}{2}} = N\mu > \delta + \|A\|, \end{aligned}$$

which contradicts to (2.6). \square

REMARK 2.3. Theorem 2.1 and Theorem 2.2 can be generalized for \mathbb{R}^n ($n \in \mathbb{N}$) as follows.

(1) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a mapping satisfying the Cauchy equation

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. If T is bounded on a small disk $B(\mathbf{x}, r) = \{\mathbf{z} \in \mathbb{R}^n \mid |\mathbf{z} - \mathbf{x}| < r\}$ for some $\mathbf{x} \in \mathbb{R}^n$, then there exists an $n \times n$ matrix A such that $T(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$. Therefore, the mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear mapping.

(2) For any $\varepsilon > 0$, there exists a mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying

$$|f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon(|\mathbf{x}| + |\mathbf{y}|)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that there is no constant $\delta \geq 0$ and no additive function $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the condition

$$|f(\mathbf{x}) - T(\mathbf{x})| \leq \delta|\mathbf{x}|$$

for all $\mathbf{x} \in \mathbb{R}^n$. So, we can say about the instability of the Cauchy equation in \mathbb{R}^n .

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